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TIME-HARMONIC PROBLEM FOR A NON-HOMOGENEOUS HALF-SPACE WITH EXPONENTIALLY VARYING SHEAR MODULUS

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Abstract—Previously published analytical solutions of dynamic problems for continuously non-homogeneous bases concerned one-dimensional, plane or axisymmetric cases. In this paper a solution in cylindrical co-ordinates is presented for an arbitrary angle distribution in the horizontal plane. The medium is assumed as isotropic, continuously non-homogeneous in the depth direction and homogeneous in the horizontal direction. Poisson's ratio is adopted as constant. For each angle component of the solution including $\cos(n\vartheta)$ or $\sin(n\vartheta)$, the problem is reduced to three ordinary differential equations (or two for the axisymmetric case where $n = 0$); two of them are coupled. Corresponding boundary conditions are formulated for given stresses or displacements at planes $z = \text{const}$. An example of non-homogeneity where shear modulus increases exponentially with depth, $G(z) = G(0)\exp(z/z_0)$, is considered (z_0 is a constant). The solution for the half-space subjected to a surface load is represented in the form of integrals including Bessel functions and suitable solutions of above-mentioned ordinary differential equations. At low frequencies the integrands have no singularities on the real axis of the complex plane; then, beginning from a definite value of the frequency (cutoff frequency), poles of integrands appear on the real axis and energy can be passed to the half-space. At some frequencies (resonance frequencies) there are double poles on the real axis leading to infinite amplitudes in the non-dissipative case. For calculations, shear modulus was treated as a complex quantity ($G(0) = G_0(1+i\varepsilon)$), where ε is a small positive constant. Results of calculations for surface displacements induced by vertical and horizontal acting point forces on the surface of the half-space are presented for static and dynamic problems, and comparison with results for the homogeneous half-space is demonstrated. © 1997 Elsevier Science Ltd.

INTRODUCTION

Known analytical solutions concerning vibration of a continuously inhomogeneous medium deal with one-dimensional, plane or axisymmetric problems. Difficulties, arising in inhomogeneous elastodynamics, result from the fact that the classical method of separation by using Helmholtz potentials is applicable only in a few cases of inhomogeneity (Hook, 1961, 1962; Alverson *et al.*, 1963). For such a separable case, in which the variations of elastic parameters and density are identical (as the square of depth), Karlsson and Hook (1963) solved the plane Lamb's problem. Separation can be achieved also in the case of incompressible medium with a linearly varying shear modulus; the corresponding solution of the axisymmetric problem was constructed by Awojobi (1972, 1973) and applied in his papers for approximate solving of a rigid disk problem for the case when shear modulus at the surface of the half-space is equal to zero. Rao (1967, 1970), Rao and Goda (1978), Vrettos (1990a, 1990b, 1991) have shown that in some cases the solution of time-harmonic problem can be obtained for coupled equations using a suitable change of the depth variable and Frobenius' method. Rao suggested a kind of non-homogeneity with the exponential variation of shear modulus and density from a value at the surface of the half-space to a limited value at infinite depth. Vrettos, considering such a type of half-space (however with constant density), unlike Rao (who used governing equations in stresses), formulated the problem in displacements which simplifies the solution. In the present paper the problem is also formulated in displacements. The general solution in cylindrical co-ordinates is constructed for a medium which is continuously non-homogeneous in the depth direction and possesses non-varying properties in the horizontal direction. As an example of non-homogeneity, the exponential law for the shear modulus increasing with depth without limit is considered. Note that an analogous medium was studied by Wilson (1942) who considered the propagation of Love waves in a half-space; Ghosh (1971) studied antiplane shear vibrations of the half-space in the case when the shear wave velocity decreases with depth.

We use similar substitution for the depth co-ordinate z as in papers by Rao, Goda and Vrettos, i.e., $\xi = \exp(-z/z_0)$. The action of vertical and horizontal time-harmonic forces on the surface of the corresponding half-space is studied with the help of Hankel's transformation and Frobenius' method which is applied to ordinary differential equations for functions entering Hankel's transforms. Dynamic behavior of the considered half-space differs significantly from the dynamic behaviour of Rao's half-space. Thus the cutoff frequency as well as resonant frequencies occur in our case whereas these phenomena are absent in the case of Rao. For avoiding the difficulties connected with singularities in the integral representation of the solution, the shear modulus is considered as complex (the half-space is treated as visco-elastic). This enables us to carry out the integration over the real axis.

BASIC EQUATIONS

The equations of motion in cylindrical co-ordinates for a linear-elastic isotropic half-space read

$$\rho\frac{\partial^2 u_r}{\partial t^2} = \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\vartheta\vartheta}}{r}$$

$$\rho\frac{\partial^2 u_\vartheta}{\partial t^2} = \frac{\partial\sigma_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{\vartheta z}}{\partial z} + \frac{2}{r}\sigma_{r\vartheta}$$

$$\rho\frac{\partial^2 u_z}{\partial t^2} = \frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\vartheta z}}{\partial\vartheta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} \tag{1}$$

where ρ is the density, $\sigma_{ij}(i,j = r,\vartheta,z)$ are the components of the stress-tensor; u_r, u_ϑ, u_z denote displacements in the directions of the co-ordinate lines; and t is time. Using Hooke's law and the expressions for the components of the strain-tensor results in

$$\sigma_{rr} = \lambda e+2G\frac{\partial u_r}{\partial r},\quad \sigma_{\vartheta\vartheta} = \lambda e+2G\left(\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{u_r}{r}\right),\quad \sigma_{zz} = \lambda e+2G\frac{\partial u_z}{\partial z},$$

$$\sigma_{r\vartheta} = G\left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta}+\frac{\partial u_\vartheta}{\partial r}-\frac{u_\vartheta}{r}\right),\quad \sigma_{\vartheta z} = G\left(\frac{\partial u_\vartheta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right),\quad \sigma_{zr} = G\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),$$

$$e = \frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z},\quad \lambda = \frac{2G\nu}{1-2\nu} \tag{2}$$

where e is the dilatation, λ is Lame's coefficient, G is the shear modulus, and ν is Poisson's ratio. For the harmonic motion with the time-dependence in the form $\exp(i\omega t)$, assuming that the shear modulus depends only on the co-ordinate z and Poisson's ratio is constant, we obtain the following equations for amplitudes (we keep for amplitudes the same notation as for displacements):

$$G\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\vartheta}{\partial\vartheta}\right)+\frac{G}{1-2\nu}\frac{\partial e}{\partial r}+\frac{\partial G}{\partial z}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\rho\omega^2 u_r = 0$$

$$G\left(\nabla^2 u_\vartheta-\frac{u_\vartheta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\vartheta}\right)+\frac{G}{1-2\nu}\frac{1}{r}\frac{\partial e}{\partial\vartheta}+\frac{\partial G}{\partial z}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\vartheta}\right)+\rho\omega^2 u_\vartheta = 0$$

$$G\nabla^2 u_z+\frac{G}{1-2\nu}\frac{\partial e}{\partial z}+2\frac{\partial G}{\partial z}\left(\frac{\partial u_z}{\partial z}+\frac{\nu e}{1-2\nu}\right)+\rho\omega^2 u_z = 0 \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2}.$$

It is assumed at this stage that $v < 1/2$; when $v \to 1/2$, the terms having the value $1-2v$ in denominators remain limited because of the fact that the dilatation, e, tends to zero. Keeping in mind to apply for the solution of eqns (3) Hankel's transformation with respect to r and Fourier's series with respect to ϑ we construct a particular solution of eqns (3) as a sum of the two following solutions:

$$U_r^{(1)} = -p(z,k)\frac{n}{\chi}J_n(\chi)\Gamma_1, \quad U_\theta^{(1)} = p(z,k)\left[J_{n-1}(\chi) - \frac{n}{\chi}J_n(\chi)\right]\Gamma_2, \quad U_z^{(1)} = 0$$

$$U_r^{(2)} = -q(z,k)\left[J_{n-1}(\chi) - \frac{n}{\chi}J_n(\chi)\right]\Gamma_1, \quad U_\theta^{(2)} = q(z,k)\frac{n}{\chi}J_n(\chi)\Gamma_2, \quad U_z^{(2)} = w(z,k)J_n(\chi)\Gamma_1 \tag{4}$$

where

$$\Gamma_1 = \begin{bmatrix}\cos(n\vartheta)\\ \sin(n\vartheta)\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix}\sin(n\vartheta)\\ -\cos(n\vartheta)\end{bmatrix}, \quad \chi = kr, \quad n = 0,1,2,\ldots.$$

In eqns (4) Bessel functions and three unknown functions $p(z,k)$, $q(z,k)$, $w(z,k)$ are introduced. The parameter k is considered as the parameter of Hankel's transformation; the sum of these two solutions must be integrated with respect to k from zero to infinity (with the corresponding treating points of singularities). Note that the similar forms of solutions with exponential functions for $p(z,k)$, $q(z,k)$, $w(z,k)$ are widely used for homogeneous half-spaces and layers (see, for example, Wolf, 1985). For $n = 0$ the first solution relates to torsional vibration, and the second solution corresponds to axisymmetric vibration. A representation similar to expressions (4) was used by Waas *et al.* (1985) for constructing solutions corresponding to free vibrations of transversely-isotropic media.

Firstly, using known relationships for Bessel functions (Abramowitz and Stegun, 1964) we write results for some operators in eqns (3) acting on $J_{n-1}(\chi)$ and $J_n(\chi)/\chi$:

$$\frac{\partial J_{n-1}(\chi)}{\partial r} = k\left(\frac{n-1}{\chi}J_{n-1}(\chi) - J_n(\chi)\right)$$

$$\frac{\partial}{\partial r}\left[\frac{J_n(\chi)}{\chi}\right] = \frac{k}{\chi}\left(J_{n-1}(\chi) - \frac{n+1}{\chi}J_n(\chi)\right)$$

$$\frac{\partial}{\partial r}\left[\frac{J_{n-1}(\chi)}{r}\right] = \frac{k^2}{\chi}\left(\frac{n-2}{\chi}J_{n-1}(\chi) - J_n(\chi)\right)$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{J_n(\chi)}{\chi}\right] = \frac{k^2}{\chi^2}\left(J_{n-1}(\chi) - \frac{n+2}{\chi}J_n(\chi)\right)$$

$$\frac{\partial^2 J_{n-1}(\chi)}{\partial r^2} = k^2\left[\frac{(n-1)(n-2)}{\chi^2}J_{n-1}(\chi) - J_{n-1}(\chi) + \frac{J_n(\chi)}{\chi}\right]$$

$$\frac{\partial^2}{\partial r^2}\left[\frac{J_n(\chi)}{\chi}\right] = \frac{k^2}{\chi}\left[\frac{(n+1)(n+2)}{\chi^2}J_n(\chi) - J_n(\chi) - \frac{3}{\chi}J_{n-1}(\chi)\right]. \tag{5}$$

Using these expressions we obtain, after substituting the solutions (4) in eqns (3), Bessel functions with indexes n and $n-1$ only. Trigonometric functions $\cos(n\vartheta)$ or $\sin(n\vartheta)$ will be as a common multiplier for all members of an equation.

Consider the first solution (4). After its substituting in the first eqn (3) the following functions of the variable r appear: $J_{n-1}(\chi)/\chi^2$, $J_n(\chi)/\chi^3$ and $J_n(\chi)/\chi$. The members containing the two first functions annihilate each other; grouping together the members with the last function leads to the following equation:

$$G\frac{d^2p}{dz^2}+\frac{dG}{dz}\frac{dp}{dz}+(\rho\omega^2-k^2G)p = 0. \tag{6}$$

After substituting the first solution (4) in the second eqn (3), in addition, we have members with $J_{n-1}(\chi)$, that generate the same eqn (6) as the members with $J_n(\chi)/\chi$, whereas another member cancels out. In the third eqn (3) all the members annihilate each other when substituting the first solution.

Analogously we treat the second solution (4). By its substituting in the first and second eqns (3) we obtain the following equation:

$$G\frac{d^2q}{dz^2}+\frac{dG}{dz}\frac{dq}{dz}+\left(\rho\omega^2-\frac{Gk^2}{\tau^2}\right)q-G\frac{1-\tau^2}{\tau^2}k\frac{dw}{dz}-\frac{dG}{dz}kw = 0 \tag{7}$$

where

$$\tau^2 = \frac{1-2\nu}{2(1-\nu)} = \frac{G}{\lambda+2G} = \frac{C_p^2}{C_s^2}. \tag{8}$$

Here C_p and C_s are velocities of the compression and shear waves respectively; this relation is constant for the considered case of non-homogeneity. Substituting the second solution (4) in the third eqn (3) leads to the following equation:

$$G\frac{d^2w}{dz^2}+\frac{dG}{dz}\frac{dw}{dz}+\tau^2(\rho\omega^2-Gk^2)w+G(1-\tau^2)k\frac{dq}{dz}+\frac{dG}{dz}(1-2\tau^2)kq = 0. \tag{9}$$

Equations (6), (7), (9) with suitable boundary conditions determine the desired functions $p(z,k)$, $q(z,k)$, $w(z,k)$; it is worthy of attention that the obtained equations are independent of n.

Consider stresses that can be needed for fitting the boundary conditions on planes $z=\text{const}$. For the sum of two solutions (4) we obtain:

$$S_{zz} = \hat{S}_{zz}\Gamma_1,\quad S_{rz} = \hat{S}_{rz}\Gamma_1,\quad S_{\vartheta z} = \hat{S}_{\vartheta z}\Gamma_2$$

$$\hat{S}_{zz} = \frac{G}{\tau^2}\left[\frac{dw}{dz}+qk(1-2\tau^2)\right]J_n(\chi)$$

$$\hat{S}_{rz} = G\left[\left(kw-\frac{dq}{dz}\right)J_{n-1}(\chi)+\left(-kw+\frac{dq}{dz}-\frac{dp}{dz}\right)\frac{nJ_n(\chi)}{\chi}\right]$$

$$\hat{S}_{\vartheta z} = G\left[\frac{dp}{dz}J_{n-1}(\chi)+\left(-kw+\frac{dq}{dz}-\frac{dp}{dz}\right)\frac{nJ_n(\chi)}{\chi}\right]. \tag{10}$$

The last two equations lead to

$$\hat{S}_{rz}+\hat{S}_{\vartheta z} = G\left(-kw+\frac{\mathrm{d}q}{\mathrm{d}z}-\frac{\mathrm{d}p}{\mathrm{d}z}\right)J_{n+1}(\chi)$$

$$\hat{S}_{rz}-\hat{S}_{\vartheta z} = G\left(kw-\frac{\mathrm{d}q}{\mathrm{d}z}-\frac{\mathrm{d}p}{\mathrm{d}z}\right)J_{n-1}(\chi). \tag{11}$$

Analogously, for the displacements corresponding to the sum of solutions (4) we can write

$$U_r = \hat{U}_r\Gamma_1,\quad U_\vartheta = \hat{U}_\vartheta\Gamma_2,\quad U_z = \hat{U}_z\Gamma_1$$

$$\hat{U}_r+\hat{U}_\vartheta = -(p-q)J_{n+1}(\chi),\quad \hat{U}_r-\hat{U}_\vartheta = -(p+q)J_{n-1}(\chi),\quad \hat{U}_z = wJ_n(\chi). \tag{12}$$

The expression for $\hat{S}_{zz}$ in (10) together with eqns (11) allow us to formulate boundary conditions for functions p, q, w through stresses, whereas expressions (12) are used for boundary conditions in displacements. Suppose for $z = z_0$ amplitudes of stresses are:

$$\sigma_{zz} = \hat{\sigma}_{zz}(r)\Gamma_1,\quad \sigma_{rz} = \hat{\sigma}_{rz}(r)\Gamma_1,\quad \sigma_{\vartheta z} = \hat{\sigma}_{\vartheta z}(r)\Gamma_2. \tag{13}$$

Integrating the sum of two solutions (4) over k and using properties of Hankel's transformation, we obtain, in accordance with eqns (10) and (11), for $z = z_0$

$$\frac{\mathrm{d}w}{\mathrm{d}z}+qk(1-2\tau^2) = \frac{k\tau^2}{G(z_0)}\int_0^\infty r\hat{\sigma}_{zz}(r)J_n(kr)\,\mathrm{d}r$$

$$-kw+\frac{\mathrm{d}q}{\mathrm{d}z}-\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{k}{G(z_0)}\int_0^\infty r[\hat{\sigma}_{rz}(r)+\hat{\sigma}_{\vartheta z}(r)]J_{n+1}(kr)\,\mathrm{d}r$$

$$kw-\frac{\mathrm{d}q}{\mathrm{d}z}-\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{k}{G(z_0)}\int_0^\infty r[\hat{\sigma}_{rz}(r)-\hat{\sigma}_{\vartheta z}(r)]J_{n-1}(kr)\,\mathrm{d}r. \tag{14}$$

The last two equations result in

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{k}{2G(z_0)}\left[\int_0^\infty r[\hat{\sigma}_{rz}(r)+\hat{\sigma}_{\vartheta z}(r)]J_{n+1}(kr)\,\mathrm{d}r+\int_0^\infty r[\hat{\sigma}_{rz}(r)-\hat{\sigma}_{\vartheta z}(r)]J_{n-1}(kr)\,\mathrm{d}r\right]$$

$$\frac{\mathrm{d}q}{\mathrm{d}z}-kw = \frac{k}{2G(z_0)}\left[\int_0^\infty r[\hat{\sigma}_{rz}(r)+\hat{\sigma}_{\vartheta z}(r)]J_{n+1}(kr)\,\mathrm{d}r-\int_0^\infty r[\hat{\sigma}_{rz}(r)-\hat{\sigma}_{\vartheta z}(r)]J_{n-1}(kr)\,\mathrm{d}r\right]. \tag{15}$$

These equations together with the first eqn (14) represent the boundary condition in the case of given stresses.

Relationships (12) can be treated analogously and lead to the boundary conditions for the functions p, q, w when displacements are given. Let there be displacements at a plane $z = z_0$ given by:

$$u_r = \hat{u}_r(r)\Gamma_1,\quad u_\vartheta = \hat{u}_\vartheta(r)\Gamma_2,\quad u_z = \hat{u}_z(r)\Gamma_1. \tag{16}$$

Analogously to eqns (14) one can obtain for $z = z_0$

$$q-p = k\int_0^{\infty} r[\hat{u}_r(r)+\hat{u}_\vartheta(r)]J_{n+1}(kr)\,\mathrm{d}r$$

$$q+p = -k\int_0^{\infty} r[\hat{u}_r(r)-\hat{u}_\vartheta(r)]J_{n-1}(kr)\,\mathrm{d}r$$

$$w = k\int_0^{\infty} r\hat{u}_z(r)J_n(kr)\,\mathrm{d}r. \tag{17}$$

The two first equations result in

$$q = \frac{k}{2}\left\{\int_0^{\infty} r[\hat{u}_r(r)+\hat{u}_\vartheta(r)]J_{n+1}(kr)\,\mathrm{d}r - \int_0^{\infty} r[\hat{u}_r(r)-\hat{u}_\vartheta(r)]J_{n-1}(kr)\,\mathrm{d}r\right\}$$

$$p = -\frac{k}{2}\left\{\int_0^{\infty} r[\hat{u}_r(r)+\hat{u}_\vartheta(r)]J_{n+1}(kr)\,\mathrm{d}r + \int_0^{\infty} r[\hat{u}_r(r)-\hat{u}_\vartheta(r)]J_{n-1}(kr)\,\mathrm{d}r\right\}. \tag{18}$$

Equations (18) together with the third eqn (17) are the desired boundary conditions for the considered case. As seen we have for the function p the independent eqn (6) and the independent boundary conditions: the last eqn (18) or the first eqn (15). Thus, the two solutions in eqns (4) can be considered as totally independent solutions.

Consider, as an example, the action of a vertical force with amplitude P_0 uniformly distributed over a circular area of radius R on the surface ($z = 0$) of a half-space. Using $n = 0$ and the upper line in Γ_j ($j = 1, 2$), we can consider only the second solution in eqns (4). The boundary condition at the plane $z = 0$ for the function w and q will be as follows:

$$\frac{\mathrm{d}w}{\mathrm{d}z} + qk(1-2\tau^2) = -\frac{k\tau^2 P_0}{G(0)\pi R^2}\int_0^R rJ_0(kr)\,\mathrm{d}r = -\frac{\tau^2 P_0}{G(0)\pi R}J_1(kR)$$

$$\frac{\mathrm{d}q}{\mathrm{d}z} - kw = 0. \tag{19}$$

To these relationships the condition of absence of sources at infinity ($z \to \infty$) must be added. Let q_1, w_1 and q_2, w_2 be two linearly independent solutions of eqns (7), (9) which satisfy corresponding boundary conditions at infinity. For example, in the case of homogeneity one can obtain on the basis of eqns (7), (9)

$$q_1 = \alpha_1 \exp(-\alpha_1 z), \quad w_1 = k\exp(-\alpha_1 z),$$

$$q_2 = k\exp(-\alpha_2 z), \quad w_2 = \alpha_2 \exp(-\alpha_z z) \tag{20}$$

where

$$\alpha_1 = (k^2 - \omega^2\rho/G)^{1/2}, \quad \alpha_2 = (k^2 - \tau^2\omega^2\rho/G)^{1/2}.$$

Introducing two arbitrary coefficients $A_1(k)$ and $A_2(k)$ and using a linear combination of two considered solutions we obtain according to (19) the following equations for these coefficients:

$$c_{11}A_1 + c_{12}A_2 = d_1, \quad c_{21}A_1 + c_{22}A_2 = d_2$$

$$c_{1j} = \frac{\mathrm{d}w_j(0,k)}{\mathrm{d}z} + k(1-2\tau^2)q_j(0,k), \quad c_{2j} = kw_j(0,k) - \frac{\mathrm{d}q_j(0,k)}{\mathrm{d}z},$$

$$d_1 = -\frac{\tau^2 P_0}{G(0)\pi R} J_1(kR), \quad d_2 = 0. \tag{21}$$

The amplitude of displacements are expressed through the integrals by parameter k (see eqns (4)):

$$u_r = \int_0^\infty [A_1 q_1(z,k) + A_2 q_2(z,k)] J_1(kr)\,\mathrm{d}k$$

$$u_z = \int_0^\infty [A_1 w_1(z,k) + A_2 w_2(z,k)] J_0(kr)\,\mathrm{d}k, \quad u_\vartheta = 0. \tag{22}$$

As a second example consider the action of a horizontal force with the amplitude Q_0 uniformly distributed over a circle of radius R on the surface ($z = 0$) of a half-space. We use: $n = 1$ (the upper line in Γ_j), $\hat{\sigma}_{zz} = 0$, $\hat{\sigma}_{rz} = -\hat{\sigma}_{\vartheta z} = -Q_0/(\pi R^2)$ for $r < R$ and zero values for all the stresses for $r > R$. From eqns (15), (14) we obtain the following boundary conditions at the plane $z = 0$:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{kQ_0}{G(0)\pi R^2}\int_0^R rJ_0(kr)\,\mathrm{d}r = \frac{Q_0}{G(0)\pi R}J_1(kR)$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} + qk(1-2\tau^2) = 0, \quad kw - \frac{\mathrm{d}q}{\mathrm{d}z} = -\frac{Q_0}{G(0)\pi R}J_1(kR). \tag{23}$$

The part of the total solution including the functions q_1, w_1, q_2, w_2 is treated as in the case of the vertical force using the coefficients A_1 and A_2; system (21) remains valid with the following changes in the right sides

$$d_1 = 0, \quad d_2 = -\frac{Q_0}{G(0)\pi R}J_1(kR). \tag{24}$$

For constructing the part of the solution containing the function $p(z,k)$ (see eqns (4)), we consider some particular solution p_1 which satisfies eqn (6) and the condition at infinity. For example, this solution for the homogeneous half-space can be accepted as $\exp(-\alpha_1 z)$. Representing the function p in the form $p = C_1 p_1$ and using the first condition (23) leads to the following expression for the coefficient C_1:

$$C_1 = -\frac{d_2}{F}\left(F = \frac{\mathrm{d}p_1(0,k)}{\mathrm{d}z}\right). \tag{25}$$

Amplitudes of displacements can be expressed using eqns (4)

$$u_r = \hat{u}_r \cos\vartheta, \quad u_\vartheta = \hat{u}_\vartheta \sin\vartheta, \quad u_z = \hat{u}_z \cos\vartheta$$

$$\hat{u}_r = \int_0^\infty \left[[q(z,k) - p(z,k)]\frac{J_1(kr)}{kr} - q(z,k)J_0(kr)\right]\mathrm{d}k$$

$$\hat{u}_\vartheta = \int_0^\infty \left[[q(z,k) - p(z,k)]\frac{J_1(kr)}{kr} + p(z,k)J_0(kr)\right]\mathrm{d}k$$

$$\hat{u}_z = \int_0^\infty w(z,k)J_1(kr)\,\mathrm{d}k \tag{26}$$

where

$$p = C_1 p_1, \quad q = A_1 q_1 + A_2 q_2, \quad w = A_1 w_1 + A_2 w_2.$$

Consider another version of the governing eqns (6), (7), (9). Note that the dilatation for the sum of solutions (4) has the form

$$e = \left(kq + \frac{\mathrm{d}w}{\mathrm{d}z} \right) J_n(kr) \Gamma_1. \tag{27}$$

Let us introduce the following two functions

$$\bar{e} = \frac{G(z)}{\tau^2} \left(kq + \frac{\mathrm{d}w}{\mathrm{d}z} \right), \quad \bar{\zeta} = G(z) \left(kw + \frac{\mathrm{d}q}{\mathrm{d}z} \right). \tag{28}$$

The function $\bar{e}$ remains limited when $\tau \to 0$ (for non-compressible medium); the function $\bar{\zeta}$ is associated with the curl of the displacement field. Equations (7) and (9) can be rewritten using the function $\bar{e}$ and $\bar{\zeta}$; together with eqns (28) this results in the following system of first-order differential equations

$$\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}z} = k\bar{e} + 2k \frac{\mathrm{d}G}{\mathrm{d}z} w - p\omega^2 q$$

$$\frac{\mathrm{d}\bar{e}}{\mathrm{d}z} = k\bar{\zeta} + 2k \frac{\mathrm{d}G}{\mathrm{d}z} q - \rho\omega^2 w$$

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\bar{\zeta}}{G} - kw$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \tau^2 \frac{\bar{e}}{G} - kq. \tag{29}$$

The boundary conditions corresponding to the first eqn (14) and the second eqn (15) will be

$$\bar{e} - 2kGq = k \int_0^{\infty} r\hat{\sigma}_{zz}(r) J_n(kr)\, \mathrm{d}r$$

$$\bar{\zeta} - 2kGw = \frac{k}{2} \left[\int_0^{\infty} r[\hat{\sigma}_{rz}(r) + \hat{\sigma}_{\vartheta z}(r)] J_{n+1}(kr)\, \mathrm{d}r - \int_0^{\infty} r[\hat{\sigma}_{rz}(r) - \hat{\sigma}_{\vartheta z}(r)] J_{n-1}(kr)\, \mathrm{d}r \right]. \tag{30}$$

The formulation (29), (30) permits calculations for the compressible medium ($\tau \neq 0$) as well as for the non-compressible medium ($\tau = 0$).

VIBRATION OF A HALF-SPACE WITH EXPONENTIAL INCREASE OF SHEAR MODULUS WITH DEPTH

Consider a non-homogeneous half-space whose shear modulus varies with depth as follows:

$$G(z) = G(0) \exp(z/z_0) \tag{31}$$

where z_0 is a some length, $G(0)$ is the shear modulus at the surface of the half-space. It is appropriate to introduce into the solution dissipative properties of the materials. For the harmonic motion one can consider a shear modulus as a complex quantity. Following the widely used approach we adopt

$$G(0) = G_0(1+\mathrm{i}\varepsilon) \tag{32}$$

where ε is a small positive constant determining measure of internal damping, and G_0 is the shear modulus at the surface for the non-dissipative case. Poisson's ratio is assumed to be real.

Action of a vertical force

This case is represented by eqns (7), (9) and conditions (19), (21). Using, for eqns (7), (9), the substitution of $\xi = \exp(-z/z_0)$ leads to

$$\xi^2\frac{\mathrm{d}^2 q}{\mathrm{d}\xi^2}+\left(\theta^2\beta^2\xi-\frac{\tilde{k}^2}{\tau^2}\right)q+\xi\frac{1-\tau^2}{\tau^2}\tilde{k}\frac{\mathrm{d}w}{\mathrm{d}\xi}-\tilde{k}w = 0$$

$$\xi^2\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2}+\tau^2(\theta^2\beta^2\xi-\tilde{k}^2)w-\xi(1-\tau^2)\tilde{k}\frac{\mathrm{d}q}{\mathrm{d}\xi}+(1-2\tau^2)\tilde{k}q = 0 \tag{33}$$

where

$$\tilde{k} = kz_0,\quad \theta = sz_0,\quad s = \omega(\rho/G_0)^{1/2},\quad \beta = (1+\mathrm{i}\varepsilon)^{-1/2}. \tag{34}$$

The complex quantity β is implied to have the positive real part. As to boundary conditions (21) we use the variable ξ and make the following changes

$$\tilde{c}_{11}A_1+\tilde{c}_{12}A_2 = \tilde{d}_1,\quad \tilde{c}_{21}A_1+\tilde{c}_{22}A_2 = \tilde{d}_2$$

$$\tilde{c}_{1j} = -\frac{\mathrm{d}w_j(1,\tilde{k})}{\mathrm{d}\xi}+\tilde{k}(1-2\tau^2)q_j(1,\tilde{k}),\quad \tilde{c}_{2j} = \tilde{k}w_j(1,\tilde{k})+\frac{\mathrm{d}q_j(1,\tilde{k})}{\mathrm{d}\xi},$$

$$\tilde{d}_j = z_0 d_j\quad (j = 1,2). \tag{35}$$

The expressions for displacements (22) can be rewritten in the form

$$u_r = -\frac{\tau^2 P_0\beta^2}{G_0\pi R}\int_0^\infty \frac{\tilde{c}_{22}q_1(\xi,\tilde{k})-\tilde{c}_{21}q_2(\xi,\tilde{k})}{D}J_1(\tilde{k}\bar{R})J_1(\tilde{k}\bar{r})\,\mathrm{d}\tilde{k}$$

$$u_z = -\frac{\tau^2 P_0\beta^2}{G_0\pi R}\int_0^\infty \frac{\tilde{c}_{22}w_1(\xi,\tilde{k})-\tilde{c}_{21}w_2(\xi,\tilde{k})}{D}J_1(\tilde{k}\bar{R})J_0(\tilde{k}\bar{r})\,\mathrm{d}\tilde{k}$$

$$D = \tilde{c}_{11}\tilde{c}_{22}-\tilde{c}_{12}\tilde{c}_{21},\quad \bar{R} = R/z_0,\quad \bar{r} = r/z_0. \tag{36}$$

Let density, ρ, be constant. Solutions of system (33) can be found in the form of power series (Frobenius' method):

$$q = \sum_{n=0}^{\infty} a_n\xi^{n+m},\quad w = \sum_{n=0}^{\infty} b_n\xi^{n+m}. \tag{37}$$

Substituting these expressions in eqns (33) and considering terms with the m-power of ξ ($n = 0$) lead to the following equations

$$\left[m(m-1)-\frac{\tilde{k}^2}{\tau^2}\right]a_0+\tilde{k}\left[\frac{1-\tau^2}{\tau^2}m-1\right]b_0 = 0$$

$$\tilde{k}[1-2\tau^2-m(1-\tau^2)]a_0+[m(m-1)-\tilde{k}^2\tau^2]b_0 = 0. \tag{38}$$

The determinant of this equation must be equal to zero which gives the equation for determination of m. This equation can be written in the form

$$g^2-2\tilde{k}^2g+\tilde{k}^4+\tilde{k}^2(1-2\tau^2)=0 \tag{39}$$

where $g=m(m-1)$. From eqn (39)

$$g_j=\tilde{k}^2-(-1)^j\mathrm{i}\tilde{k}(1-2\tau^2)^{1/2} \quad (j=1,2). \tag{40}$$

Four corresponding values of m will be

$$\begin{aligned} m_j &= 0.5+(0.25+g_j)^{1/2} \quad (j=1,2) \\ m_j &= 0.5-(0.25+g_{j-2})^{1/2} \quad (j=3,4) \end{aligned} \tag{41}$$

where the radicals having the positive real part are meant. For matching the condition for $z\to\infty$ ($\xi\to 0$) we will use the two solutions q_1, w_1 and q_2, w_2 corresponding to m_1 and m_2, respectively. For these solutions we set $a_0^{(j)}=1$ ($j=1,2$) and the corresponding values of $b_0^{(j)}$ are obtained from the second eqn (38):

$$b_0^{(j)}=\frac{m_j(1-\tau^2)+2\tau^2-1}{\tilde{k}(1-\tau^2)-(-1)^j\mathrm{i}(1-2\tau^2)^{1/2}} \quad (j=1,2). \tag{42}$$

Consequent coefficients in series (37) can be found as the result of substituting these series in eqns (33) and considering the terms containing ξ^{n+m} for $n=1, 2, \ldots$. The following recurrent system of equations is obtained:

$$\begin{aligned} &\left[(n+m_j)(n+m_j-1)-\frac{\tilde{k}}{\tau^2}\right]a_n^{(j)}+\tilde{k}\left[\frac{1-\tau^2}{\tau^2}(n+m_j)-1\right]b_n^{(j)} = -\theta^2\beta^2a_{n-1}^{(j)} \\ &\tilde{k}[1-2\tau^2-(1-\tau^2)(n+m_j)]a_n^{(j)}+[(n+m_j)(n+m_j-1)-\tilde{k}^2\tau^2]b_n^{(j)} = -\theta^2\beta^2\tau^2b_{n-1}^{(j)} \\ &\qquad (n=1,2,\ldots). \end{aligned} \tag{43}$$

Beginning from $a_0^{(j)}$, $b_0^{(j)}$ one can find the coefficients $a_n^{(j)}$, $b_n^{(j)}$ ($n=1,2,\ldots$) with the help of system (43). The convergence of series (37) is better for smaller values of the parameter θ; (for the static case only one term for $n=0$ is needed). Using the 8 bytes number format for variables, one can perform calculations with sufficient accuracy till values of θ about 20.

For the real values of $\tilde{k}$ and $\beta=1$ (non-dissipative case) all quantities associated with the second solution are complex conjugate to the corresponding quantities for the first solution. From this follows that the expressions entering into integrals (36) will be in the considered case

$$\frac{\tilde{c}_{22}q_1(\xi,\tilde{k})-\tilde{c}_{21}q_2(\xi,\tilde{k})}{D}=\frac{\mathrm{Im}\,[\tilde{c}_{21}q_2(\xi,\tilde{k})]}{\mathrm{Im}(\tilde{c}_{12}\tilde{c}_{21})}, \quad \frac{\tilde{c}_{22}w_1(\xi,\tilde{k})-\tilde{c}_{21}w_2(\xi,\tilde{k})}{D}=\frac{\mathrm{Im}\,[\tilde{c}_{21}w_2(\xi,\tilde{k})]}{\mathrm{Im}(\tilde{c}_{12}\tilde{c}_{21})}. \tag{44}$$

For small values of the parameter θ these quantities are non-singular for positive values of $\tilde{k}$ and integrals (36) can be evaluated directly. Amplitudes of displacements are real for these small frequencies; this means that energy is not transmitted to the half-space. Beginning from $\theta=\theta_0=1.2024$ the first pole of quantities (44) appears in the vicinity of the point $\tilde{k}=0$, and the proper avoiding this pole in the complex plane $\tilde{k}$ brings into existence an imaginary part for the amplitudes of the displacements. The frequency θ_0 plays the role of the so-called cutoff frequency, for $\theta>\theta_0$ the motion can propagate in the half-space.

With increase of the frequency parameter θ, additional poles come into being. The values of θ, at which new zeros of D appear, are associated with the form of solution of eqns (33) at the point $\tilde{k} = 0$ when the equations are separated

$$\xi \frac{d^2 q}{d\xi^2} + \theta^2 \beta^2 q = 0$$

$$\xi \frac{d^2 w}{d\xi^2} + \tau^2 \theta^2 \beta^2 w = 0. \tag{45}$$

Solutions of these equations are expressed through Bessel function (Abramowitz and Stegun 1964). In the non-dissipative case ($\beta = 1$), two required solutions will be as follows:

$$q_1 = \xi^{1/2} J_1(2\theta\xi^{1/2}), w_1 = 0; \quad q_2 = 0, w_2 = \xi^{1/2} J_2(2\tau\theta\xi^{1/2}). \tag{46}$$

In this case the coefficients of the system (35) have the form

$$\tilde{c}_{11} = 0, \quad \tilde{c}_{22} = 0, \quad \tilde{c}_{21} = \frac{dq_1(1)}{d\xi} = \theta J_0(2\theta), \quad \tilde{c}_{12} = -\frac{dw_2(1)}{d\xi} = -\tau\theta J_0(2\tau\theta). \tag{47}$$

Thus, zeros of D coincide with the zeros of the functions $J_0(2\theta)$ and $J_0(2\tau\theta)$, and at the corresponding non-dimensional frequencies θ the changes of the number of poles of the integrands take place. The cutoff non-dimensional frequency, θ_0, is equal to half of the first zero of the Bessel function $J_0(x)$. Note that in the case $\tilde{k} = 0$ we have the equations of one-dimensional motions of a bar, subjected to shear deformations (the first eqn (45)) and longitudinal deformations without lateral displacements (the second equation). It is clear that the found frequencies are corresponded to the resonance frequencies of the considered infinite bars. As to half-space with a bounded loading area on its surface, simple poles of integrands in (36) do not produce an infinite increase in the solution. However, the half-space in question possesses resonance frequencies owing to double roots of the denominator D. These frequencies lie slightly lower of odd zeros of the function $J_0(2\tau\theta)$; thus for $\tau = 0.5$ ($\nu = 1/3$) the three first resonance values of θ are 2.2840, 8.6316, 14.9181 (the first three above-mentioned odd zeros are 2.4048, 8.6537, 14.9309). The values of $\tilde{k}$, which correspond to the double root for these frequencies, are equal to 0.399, 0.1882, 0.1581, respectively. For illustration, in Fig. 1 the behavior of the denominator D on a part of the $\tilde{k}$-axis is shown for the first resonance frequency ($\beta = 1$).

Calculations show that singularities of the integrands can lie between $\tilde{k} = 0$ and $\tilde{k} = A\theta$ (for $\theta < 20$), where the coefficient A decreases from 1.1 (for $\nu \to 0$) to 0.98 (for $\nu \to 0.5$).

Numerical results correspond to the case of the point force ($R \to 0$); in this case the value of $J_1(\tilde{k}\tilde{R})/R$ in the integrals is replaced with $0.5\tilde{k}/z_0$. For evaluation of an integral like (36) a complex value of the parameter β was introduced which eliminates all the above-mentioned singularities. When numerically integrating, an enough small dividing the interval, where the singularities lie for the non-dissipative material, is used. The part of the integrals corresponding to the interval $B < \tilde{k} < \infty$ (where B is a large value permitting the asymptotic representation of Bessel functions in (36) and surpassing all the considered singularities) is evaluated by means of the integration by parts that gives an asymptotic expansion in negative powers of $B\tilde{r}$ (only the first so obtained member was used).

Firstly we consider the numerical results for the static problem. Using eqns (44) the corresponding solution can be written as follows ($R \to 0, \beta = 1$):

$$u_r = -\frac{P_0(1-2\nu)}{4G_0\pi r} S_{vh}, \quad S_{vh} = \frac{2\tau^2}{(1-2\nu)} \int_0^\infty \tilde{k}\tilde{r} \frac{\operatorname{Im}(\tilde{c}_{21}\xi^{m_2})}{\operatorname{Im}(\tilde{c}_{12}\tilde{c}_{21})} J_1(\tilde{k}\tilde{r})\, d\tilde{k}$$

$$u_z = \frac{P_0(1-\nu)}{2G_0\pi r} S_{vv}, \quad S_{vv} = -\frac{\tau^2}{(1-\nu)} \int_0^\infty \tilde{k}\tilde{r} \frac{\operatorname{Im}(\tilde{c}_{21} b_0^{(2)} \xi^{m_2})}{\operatorname{Im}(\tilde{c}_{12}\tilde{c}_{21})} J_0(\tilde{k}\tilde{r})\, d\tilde{k} \tag{48}$$

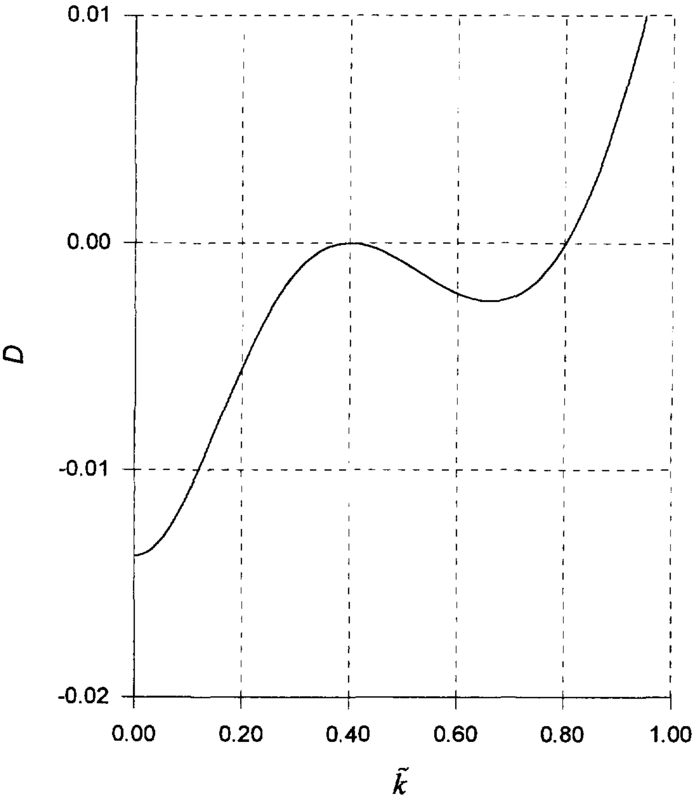


Fig. 1. Denominator, D, of integrands vs the integration parameter $\tilde{k}$ for the non-dissipative case at the first resonance frequency ($\theta = 2.284$). The double zero, leading to the resonance, corresponds to the value of $\tilde{k} = 0.399$.

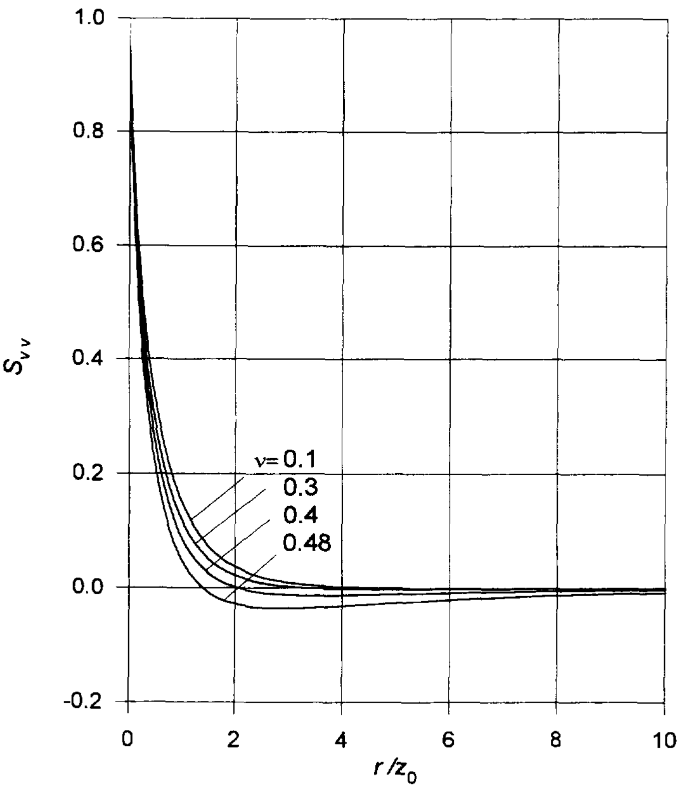
where

$$\tilde{c}_{12} = -b_0^{(2)}m_2+\tilde{k}(1-2\tau^2), \quad \tilde{c}_{21} = \tilde{k}b_0^{(1)}+m_1.$$

The non-dimensional factors S_{vh} and S_{vv} give a correction to the static solution for the homogeneous case. In Fig. 2 these factors are presented for $\xi = 1$ (the surface of the half-space) and for some values of Poisson's ratio, ν. All curves pass through the point (0, 1) corresponding to the homogeneous half-space. The displacements are strongly influenced by the parameter non-homogeneity, $\bar{r} = r/z_0$. Poisson's ratio has significant effect on horizontal displacements for materials with low compressibility; for $\nu > 0.3$–0.4 values and directions of horizontal displacements of the considered foundation differ considerably from those for the homogeneous half-space.

In the dynamic case, expressions (48) can be used with introducing the value β^2 before the integrals and with the replacement of the ratios of imaginary parts with the original expressions in (36), $[\tilde{c}_{22}q_1(\xi,\tilde{k})-\tilde{c}_{21}q_2(\xi,\tilde{k})]/D$ and $[\tilde{c}_{22}w_1(\xi,\tilde{k})-\tilde{c}_{21}w_2(\xi,\tilde{k})]/D$ for u_r and u_z, respectively. For the sake of comparison with the homogeneous case it is convenient to use, instead of $\bar{r}$, the parameter $a = \omega r(\rho/G_0)^{1/2}$ ($a = \theta\bar{r}$). Figures 3 and 4 show the behavior of the real and imaginary parts of amplitudes of vibration for ε in (32) equal to 0.01. The solution for the homogeneous half-space is represented by the dashed lines. As seen, the solutions approach to the homogeneous solution with the increase in the parameter θ. The value $\theta = 2.3$ is close to the first resonance frequency for the non-dissipative case (2.284); the behavior of the solution for this and for less values of the parameter θ is greatly dissimilar from that corresponding to the homogeneous solution.

(a)



(b)

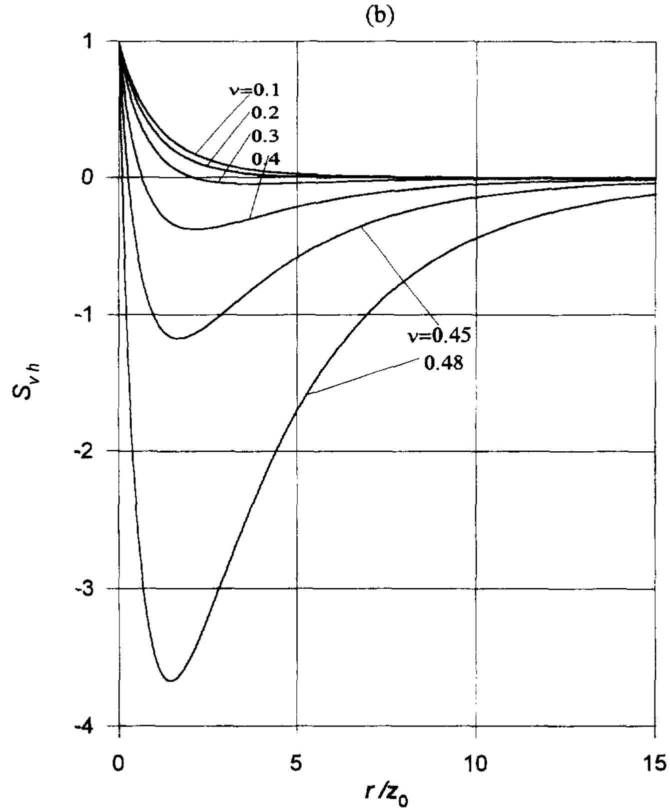
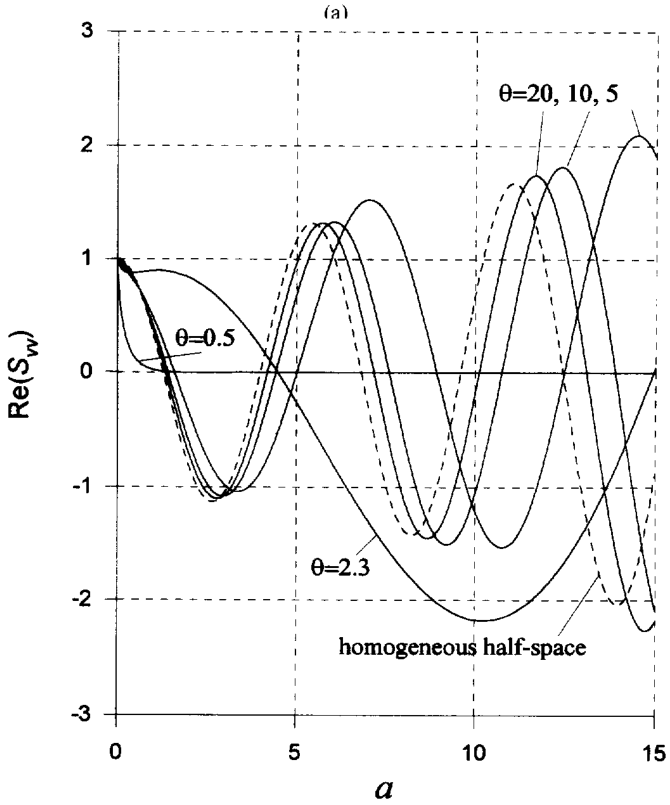


Fig. 2. Static vertical (a) and horizontal (b) normalized surface displacements of the non-homogeneous half-space, subjected to the vertical point force, for some values of Poisson's ratio.



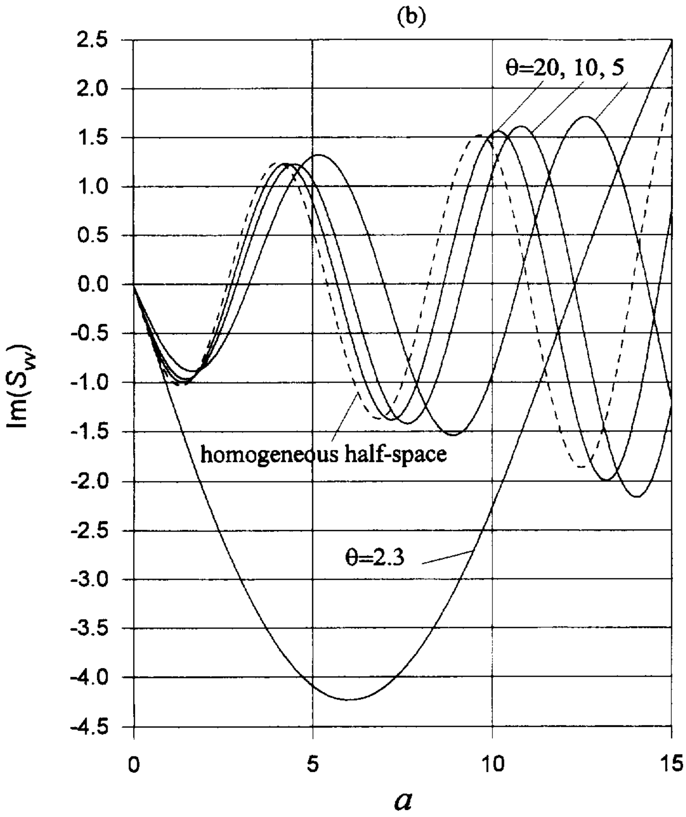


Fig. 3. Real part (a) and imaginary part (b) of normalized vertical surface displacements, caused by the surface vertical point force, vs frequency parameter a for $\nu = 1/3$, $\varepsilon = 0.01$ and for some values of the parameter θ.

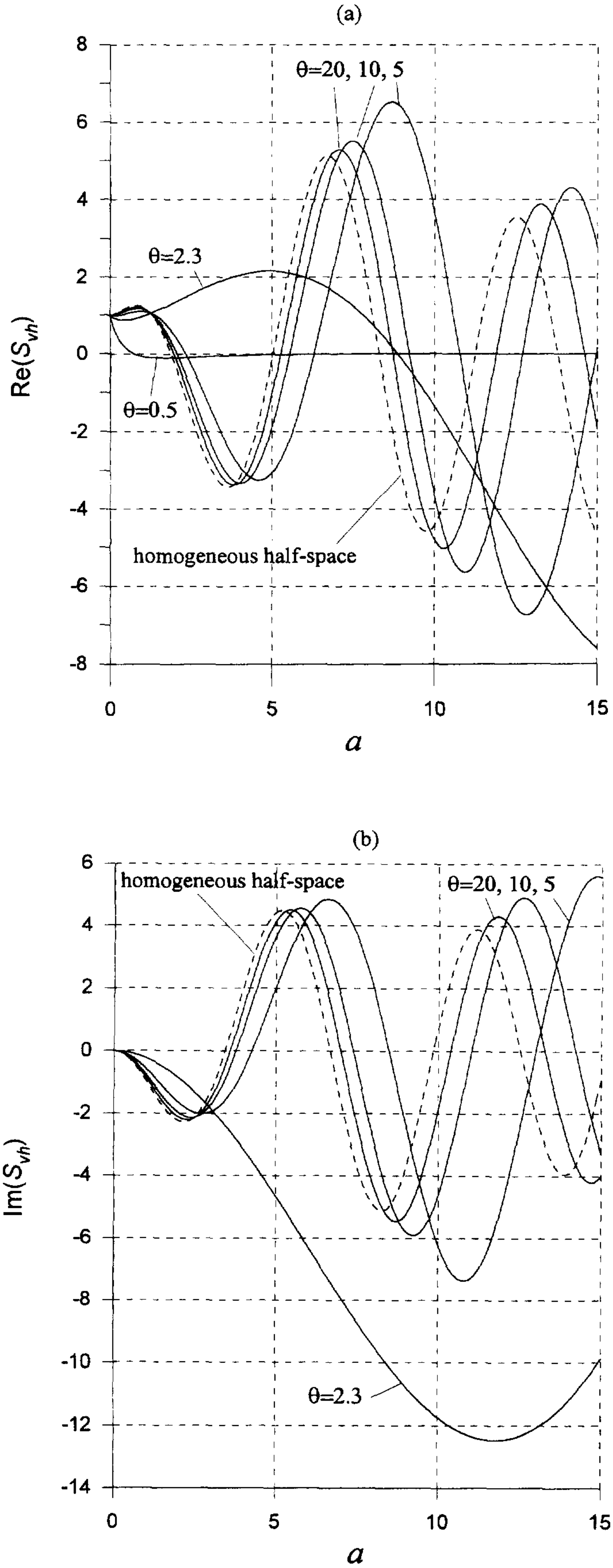


Fig. 4. Real part (a) and imaginary part (b) of normalized horizontal surface displacements, caused by a surface vertical point force, vs frequency parameter a for $\nu = 1/3$, $\varepsilon = 0.01$ and for some values of the parameter θ.

Action of a horizontal force

As the second example for the half-space with the exponentially increasing stiffness consider the action of a horizontal harmonic force. Equations (6) for the function p using the variable ξ can be rewritten in the form

$$\xi^2 \frac{d^2 p}{d\xi^2} + (\theta^2 \beta^2 \xi - \tilde{k}^2) p = 0. \tag{49}$$

A solution of this equation with the required behavior in vicinity of the point $\xi = 0$ can be represented using Bessel function in the form (Abramowitz and Stegun, 1964):

$$p(\xi, \tilde{k}) = C\xi^{1/2} J_\mu(2\theta\beta\xi) \quad (\mu = \sqrt{1+4\tilde{k}^2}) \tag{50}$$

where C is an arbitrary constant. Another way is to apply, as above, the series representation

$$p = \sum_{n=0}^{\infty} c_n \xi^{n+m} \tag{51}$$

with the following equation for m:

$$m^2 - m - \tilde{k}^2 = 0 \tag{52}$$

whose roots are as follows

$$m_{1,2} = 0.5(1 \pm \sqrt{1+4\tilde{k}^2}). \tag{53}$$

The root with the plus sign before the radical (m_1) must be chosen for fulfilling the condition at the point $\xi = 0$ ($z \to \infty$). Setting $c_0 = 1$ one can find subsequent coefficients with the help of the following recurrent relationship

$$c_n = -\frac{\beta^2 \theta^2 c_{n-1}}{n(n+2m_1-1)}. \tag{54}$$

This result corresponds to (50). Using expressions (50) or (51) (with $m = m_1$) as the solution p_1 in eqns (25) and (26) we can represent the constant C_1 as

$$C_1 = \frac{\tilde{d}_2}{\tilde{F}} \left(\tilde{F} = \frac{dp_1(1, \tilde{k})}{d\xi} \right). \tag{55}$$

Defining the coefficients A_1 and A_2 by eqns (35) with the regarding eqns (24), expressions (26) can be rewritten, analogously to (36), as follows

$$\hat{u}_r = \frac{Q_0 \beta^2}{G_0 \pi R} \int_0^\infty J_1(\tilde{k}\tilde{R}) H_r(\xi, \tilde{k})\, d\tilde{k},$$

$$H_r(\xi, \tilde{k}) = \left[\frac{\tilde{c}_{12} q_1(\xi, \tilde{k}) - \tilde{c}_{11} q_2(\xi, \tilde{k})}{D} + \frac{p_1(\xi, \tilde{k})}{\tilde{F}} \right] \frac{J_1(\tilde{k}\tilde{r})}{\tilde{k}\tilde{r}} - \frac{\tilde{c}_{12} q_1(\xi, \tilde{k}) - \tilde{c}_{11} q_2(\xi, \tilde{k})}{D} J_0(\tilde{k}\tilde{r})$$

$$\hat{u}_\vartheta = \frac{Q_0 \beta^2}{G_0 \pi R} \int_0^\infty J_1(\tilde{k}\tilde{R}) H_\vartheta(\xi, \tilde{k})\, d\tilde{k},$$

$$H_\vartheta(\xi, \tilde{k}) = \left[\frac{\tilde{c}_{12} q_1(\xi, \tilde{k}) - \tilde{c}_{11} q_2(\xi, \tilde{k})}{D} + \frac{p_1(\xi, \tilde{k})}{\tilde{F}} \right] \frac{J_1(\tilde{k}\tilde{r})}{\tilde{k}\tilde{r}} - \frac{p_1(\xi, \tilde{k})}{\tilde{F}} J_0(\tilde{k}\tilde{r})$$

$$\hat{u}_z = \frac{Q_0 \beta^2}{G_0 \pi R} \int_0^\infty J_1(\tilde{k}\tilde{R}) H_z(\xi, \tilde{k})\, d\tilde{k}, \quad H_z(\xi, \tilde{k}) = \frac{\tilde{c}_{12} w_1(\xi, \tilde{k}) - \tilde{c}_{11} w_2(\xi, \tilde{k})}{D} J_1(\tilde{k}\tilde{r}). \tag{56}$$

For the case of a point force, the solution can be represented as the static displacements at the surface of the homogeneous half-space multiplied by the corresponding factor

$$\hat{u}_r = \frac{Q_0}{2G_0\pi r}S_{hh1}, \quad S_{hh1} = \beta^2\int_0^\infty \tilde{k}\tilde{r}H_r(\xi,\tilde{k})\,\mathrm{d}\tilde{k}$$

$$\hat{u}_\vartheta = -\frac{Q_0(1-v)}{2G_0\pi r}S_{hh2}, \quad S_{hh2} = -\frac{\beta^2}{(1-v)}\int_0^\infty \tilde{k}\tilde{r}H_\vartheta(\xi,\tilde{k})\,\mathrm{d}\tilde{k}$$

$$\hat{u}_z = \frac{Q_0(1-2v)}{4G_0\pi r}S_{hv}, \quad S_{hv} = \frac{2\beta^2}{(1-2v)}\int_0^\infty \tilde{k}\tilde{r}H_z(\xi,\tilde{k})\,\mathrm{d}\tilde{k}. \tag{57}$$

In the static case ($\theta = 0, \beta = 1$), only one addendum (for $n = 0$) is kept in series (37), (51), and the property of conjugation can be used analogously to eqns (44). The static behavior of the factors S_{hh1} and S_{hh2} for $\xi = 1$ is shown in Fig. 5 for several values of Poisson's ratio. Note that in accordance with the principle of reciprocity, the quantities S_{hv} and S_{vh} must be identical for $\xi = 1$ (also in the dynamic case). This fact was checked by calculations.

For the dynamic case, the singularities (the zeros of denominator D in (36) and (56)), considered when studying the action of the vertical force remain valid. In addition, the singularities associated with zeros of the quantity $\tilde{F}$ appear when parameter θ exceed the value of θ_0 corresponding to the first zero of the function $J_0(2\theta)$. Unlike D these additional singularities are always simple poles producing an imaginary part in complex amplitudes of vibration. In Figs 6 and 7 the factors S_{hh1} and S_{hh2} are given for the surface of the half-space at some values of parameter θ; the dissipative parameter ε is equal to 0.01. In distinction to the case of the vertical point, nonhomogeneity influences more significantly the behavior of the amplitudes at large distances from the acting force. The noticeable increase of the normalized amplitudes S_{hh2} with the increase of parameter a can be explained by the influence of the waves of Love's type which are generated by the horizontal surface force.

CONCLUSIONS

In this paper, the general solution for the response of the non-homogeneous (in the depth direction) isotropic elastic medium (a half-space or a layer) to a time-harmonic surface loading with arbitrary angle distribution has been constructed. On the basis of this solution a half-space with the exponentially increasing shear modulus, subjected to vertical and horizontal point forces has been considered. The corresponding solutions have the form of infinite integrals including Bessel functions and power series in variable $\xi = \exp(-z/z_0)$ (for the static case only first members of the series remain). For small frequencies there are no singularities in integrands on the real axis of the complex plane and the integrals can be calculated immediately. Beginning from a definite value of the frequency (cutoff frequency), poles on the real axis appear; at some frequencies (resonance frequencies) the one of poles is double which leads to infinite values of amplitudes at these frequencies. These difficulties are overcome by introducing a material damping into the solution. The results obtained show that non-homogeneity can considerably influence the response of a half-space to dynamic loading especially in respect to the wave propagation pattern. Thus, the radiation damping known in the case of the homogeneous half-space can be absent or be significantly smaller for the considered half-space at low frequencies.

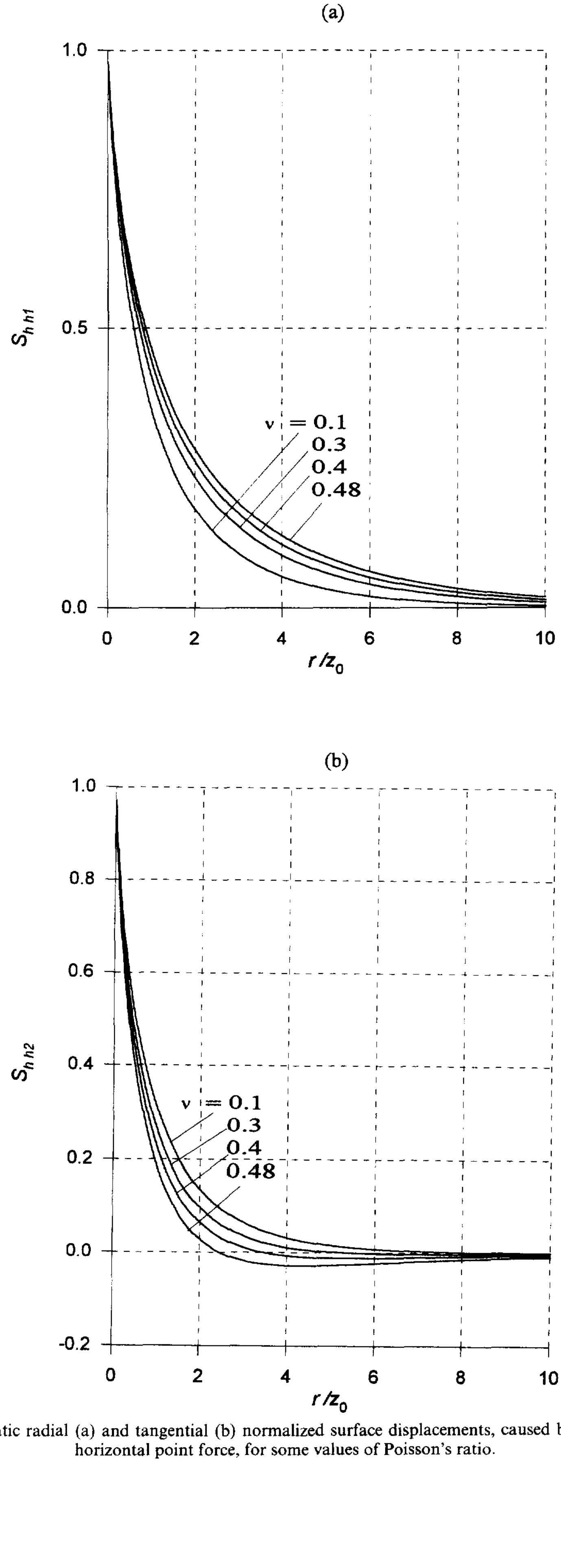


Fig. 5. Static radial (a) and tangential (b) normalized surface displacements, caused by a surface horizontal point force, for some values of Poisson's ratio.

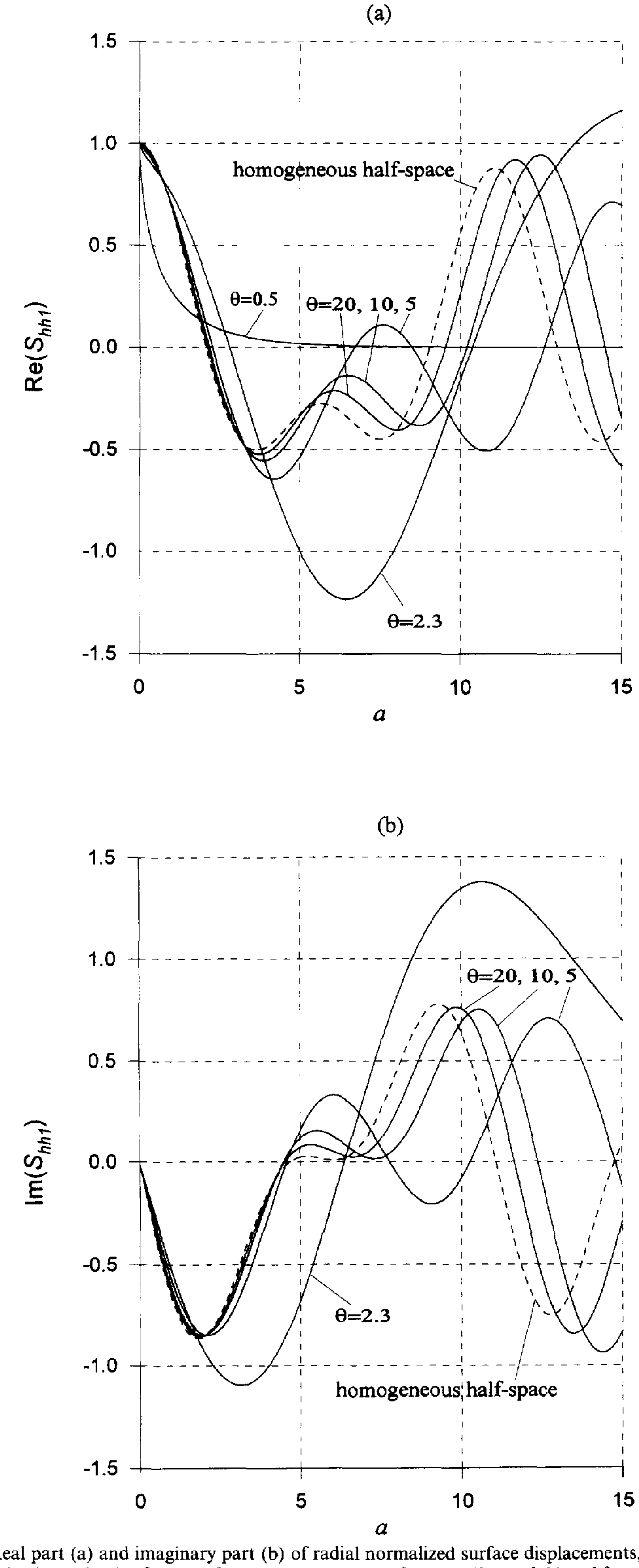
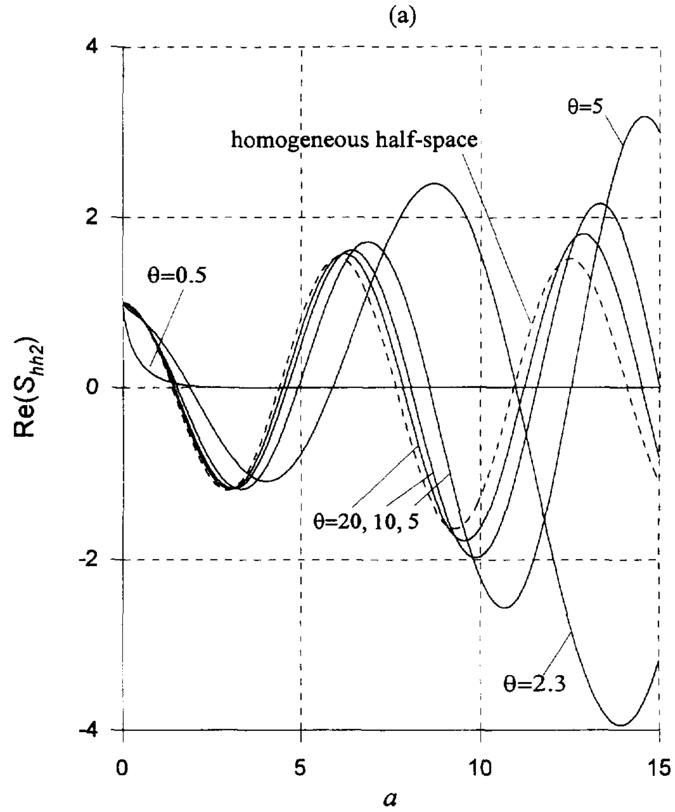


Fig. 6. Real part (a) and imaginary part (b) of radial normalized surface displacements, caused by a surface horizontal point force, vs frequency parameter a for $\nu = 1/3$, $\varepsilon = 0.01$ and for some values of the parameter θ.



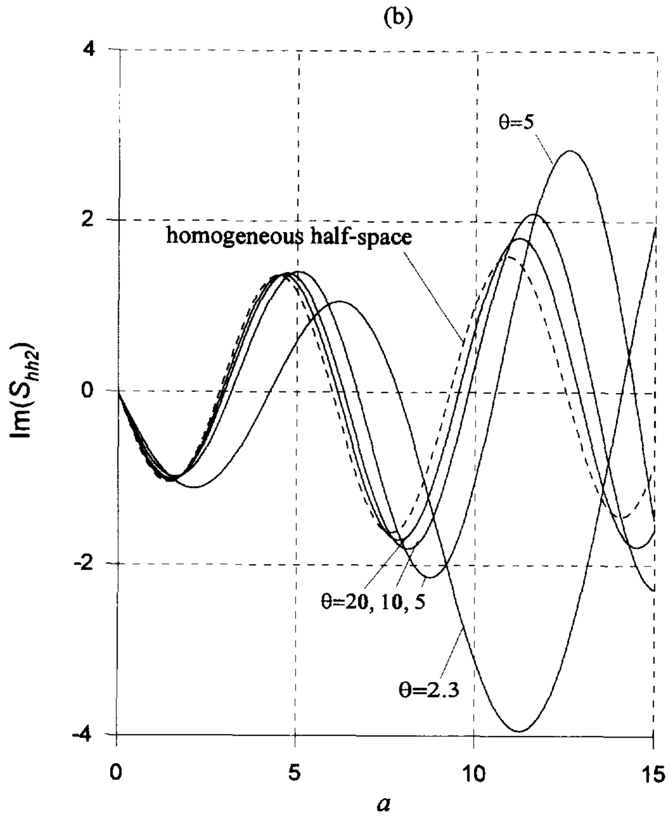


Fig. 7. Real part (a) and imaginary part (b) of tangential normalized surface displacements, caused by a surface horizontal point force, vs frequency parameter a for $\nu = 1/3$, $\varepsilon = 0.01$ and for some values of the parameter θ.

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